

# 4. ARITHMETIC AND GEOMETRIC SERIES

## §4.1. Arithmetic Sequences and Series

Can you guess the next term in the sequence 8, 13, 18, 23, ...? The way to solve such puzzles is to find the pattern and then continue it. Here it seems that the pattern is to add 5 to each term to get the next. In this case the next term will be 28.

Of course there's no guarantee that this is the pattern. It's possible to come up with a very complicated pattern that gives 8, 13, 18, 23 as the first 4 terms but something other than 28 for the fifth.

For example I could say that the pattern I had in mind was for the  $n$ 'th term to be  $n^4 - 10n^3 + 35n^2 - 45n + 27$ . The first four terms would be 8, 13, 18, 23 but the next term would be 52. Always with this sort of question the understanding is to come up with the *simplest* rule that fits the given numbers and use it to produce the next. Of course we might disagree as to what is the simplest rule.

An **arithmetic sequence** (sometimes called an arithmetic progression, or **AP** for short) is obtained by adding the terms of an arithmetic sequence. If the first term is  $a$  and we add  $d$  each time then the sequence is

$$a, a + d, a + 2d, a + 3d, \dots$$

An **arithmetic series** is where we add a finite number of consecutive terms in an arithmetic sequence:

$$a + [a + d] + [a + 2d] + [a + 3d] + \dots + [a + (n - 1)d].$$

The number  $a$  is called the **first term** and the number  $d$  is called the **common difference**. The  $n$ 'th term is often denoted by  $T_n$ , so  $T_1 = a$  and the sum of the first  $n$  terms is denoted by  $S_n$ .

**Theorem 1:** For an arithmetic series with first term  $a$ , last term  $L$  and common difference  $d$ :

$$(1) T_n = a + (n - 1)d,$$

$$(2) S_n = \frac{n}{2} [2a + (n - 1)d] .$$

$$(3) S_n = \frac{n}{2} [a + L],$$

**Proof:** (1) In going from  $T_1$  to  $T_n$  there are  $n - 1$  steps, each involving the addition of  $d$ .

Hence  $T_n = a + (n - 1)d$ .

(2)

$$S_n = T_1 + T_2 + \dots + T_{n-1} + T_n.$$

Reversing the order,

$$S_n = T_n + T_{n-1} + \dots + T_2 + T_1.$$

Adding we get  $2S_n$

$$= (T_1 + T_n) + (T_2 + T_{n-1}) + \dots + (T_{n-1} + T_2) + (T_n + T_1)$$

Now for any  $h$  we have  $T_h + T_{n+1-h}$

$$= [a + (h - 1)d] + [a + (n + 1 - h)d] = 2a + (n - 1)d.$$

So in the expression for  $2S_n$  we have  $n$  bracketed pairs, where each is  $2a + (n - 1)d$  and so:

$$2S_n = n[2a + (n - 1)d]$$

and hence (2) follows.

$$(3) \text{ From (2), } S_n = \frac{n}{2} [2a + (n - 1)d]$$

And, from (1),  $L = a + (n - 1)d$ , we have  $S_n = \frac{n}{2} [a + L]$ .



**Example 1:** Find the 100<sup>th</sup> term of the arithmetic sequence 8, 13, 18, 23, ...

**Solution:** Here  $a = 8$  and  $d = 5$ , so:

$$T_{100} = 8 + (99)(5) = 503.$$

**Example 2:** Find  $1 + 2 + 3 + \dots + 100$ .

**Solution:** Here  $a = 1$  and  $d = 1$  and  $n = L = 100$ .

$$\therefore S_{100} = \frac{100}{2} [2 + 99] = 50.101 = 5050.$$

Here the calculation is so simple that one could do it in one's head. There's a story that a young boy called Carl was in a mathematics class where the teacher wanted to occupy the pupils for a whole hour with the minimum amount of work on his part. So he wrote up the above problem, expecting that it would take most of the lesson for the boys to do the 99 additions, while he could snooze. He was most annoyed when Carl called out the answer before the teacher had even sat down. The boys hadn't

been taught the arithmetic series formula but Carl had figured it out for himself. He realised that if you add the first and the last you get 101. Similarly if you add the second and the second last you also get 101. Pairing the 100 numbers into 50 pairs in this way the answer must be 50 times 101.

Of course the teacher wasn't at all pleased and history doesn't record whether Carl was kept in after school for insolence! But he did go on and become the famous German mathematician Carl Friedrich Gauss [1777 – 1855].

**Example 3:** A staircase with 20 steps is to be made out of concrete, with the space below the stairs completely filled by concrete. The width of each stair is 1 metre, the depth of each tread is 20 cm and the height of each step is 15 cm. How many cubic metres of concrete will be needed?

**Solution:** The first step is  $100 \times 20 \times 15$  cubic centimetres ( $\text{cm}^3$ ) =  $30000 \text{ cm}^3 = 0.03$  cubic metres. The second step consists of two of these and so requires  $0.06 \text{ m}^3$ .

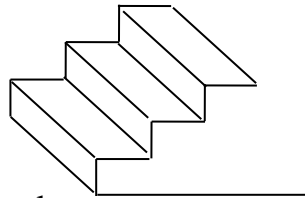
The total amount of concrete, in cubic metres, is the sum of 20 terms of the sequence

0.03, 0.06, 0.09, ...

Here  $a = 0.03 = d$ .

So  $S_{20} = 10[0.06 + 19 \times 0.03]$   
 $= 10[21 \times 0.03] = 6.3$ , so

6.3 cubic metres of concrete is required.



**Example 4:** A large roll of paper has a radius of 50cm. The inner core has a radius of 10cm. The paper has a thickness of 0.1mm. How long is the paper when unrolled?

**Solution:** The end of the roll goes in a continuous spiral, but a good approximation would be to consider it as a series of concentric circles.

The thickness is 0.01cm.

The innermost circle

would have radius 10.005

(the 0.005 being half the thickness of the paper).

The number of circles will

$$\text{be } n = \frac{50 - 10}{0.01} = 4000.$$



Using the fact that the circumference of a circle is  $2\pi r$ , where  $r$  is the radius, we see that the circumference of the innermost circle will be  $20.01\pi$ . The radius of the next circle will be 10.015 and so its length (circumference) will be  $20.03\pi$ . Each successive circle will have a radius that is 0.1 more than the previous one, and so the length will be  $0.02\pi$  more than the previous one.

The radius of the outermost circle will be  $50 - 0.05 = 49.95$  and so the length of the outermost circle will be  $2\pi \times 49.95 = 99.99\pi$ . Here we have an arithmetic sequence where  $a = 20.01\pi$  and  $d = 0.02\pi$ . The last term will be  $L = 99.99\pi$ .

So  $S_{4000} = 2000[20.01\pi + 99.99\pi] = 240000\pi$  which is approximately 753982cm. This is about 7540 metres.

Another way we could have solved this is to work out the total area of one edge of the paper. If the length, when unrolled, is  $L$  cm then the area is  $0.01L$  cm<sup>2</sup>. The area of that edge, when on the roll is the difference between the area of the outside circle and the inside circle. This is  $\pi[50^2 - 10^2] = 2400\pi$  cm<sup>2</sup>.

Hence  $0.01L = 2400\pi$  and so  $L = 240000\pi$  cm, giving the same answer as above.

I remember as a boy having an old reel to reel tape recorder – the forerunner of the audio cassette. I worked out a way of telling how much time I had left on the tape by working out the length of the remaining tape along the above lines.

## §4.2. Geometric Sequences and Series

A **geometric sequence** is one where we multiply each term by the same number to get the next. This number is called the common ratio. The sequence can therefore be written as

$$a, ar, ar^2, \dots$$

where  $a$  is the first term and  $r$  is the common ratio.

A **geometric series** is where we add a finite or infinite number of terms in a geometric sequence:

$$a + ar + ar^2 + \dots + ar^{n-1}.$$

The old phrase for geometric series is ‘geometric progression’. Although that term has dropped out you will often hear a geometric series referred to as an **AP**.

As with arithmetic series we use the symbols  $T_n$  and  $S_n$  to denote the  $n$ 'th term and the sum of the first  $n$  terms respectively.

**Theorem 2:** For a geometric series with first term  $a$  and common ratio  $r \neq 1$ :

$$(1) T_n = ar^{n-1} \text{ and}$$

$$(2) S_n = a \left( \frac{1 - r^n}{1 - r} \right).$$

**Proof:** (1) From  $T_1$  to  $T_n$  there are  $n - 1$  steps, each involving the multiplication by  $r$ .

So  $T_n = ar^{n-1}$ .

$$(2) \quad S_n = a + ar + ar^2 + \dots + ar^{n-2} + ar^{n-1}.$$

$$\text{Hence } rS_n = ar + ar^2 + ar^3 + \dots + ar^{n-1} + ar^n.$$

Subtracting we get  $(1 - r)S_n = a - ar^n$ .

$$\text{And so } S_n = a \left( \frac{1 - r^n}{1 - r} \right). \quad \text{👋 😊}$$

The above theorem leaves out what happens when  $r = 1$ . Clearly with  $1 - r$  in the denominator we can't use the above formula for  $S_n$ . But if  $r = 1$  the series becomes:

$$a + a + a + \dots + a$$

and consequently  $S_n = na$  in that case.

**Example 5:**

Find the 20<sup>th</sup> term of the series  $3 + 6 + 12 + \dots$

**Solution:** Here  $a = 3$  and  $r = 2$ , so  $T_{20} = 3 \cdot 2^{19} = 1572864$  (with the help of a calculator).

**Example 6:** Find  $3 + 6 + 12 + \dots + 1572864$ .

**Solution:** Again  $a = 3$ ,  $r = 2$ . Also,  $n = 20$ .

$$\text{Hence } S_{20} = 3 \left( \frac{2^{20} - 1}{2 - 1} \right) = 3 \cdot (2^{20} - 1) = 3145725.$$

**Example 7:** An insect travels 1 metre in the first minute, 50cm in the 2<sup>nd</sup> and 25cm in the 3<sup>rd</sup> minute. Each minute it travels half the distance as in the previous minute. How far will it have travelled after 12 minutes?

**Solution:** Using centimetres as the unit we have  $a = 100$  and  $r = \frac{1}{2}$ .

$$S_{12} = 100 \left( \frac{1 - \frac{1}{2}^{12}}{1 - \frac{1}{2}} \right) = 200(1 - \frac{1}{2}^{12}) = 199.95 \text{ (approx.)}.$$

It's clear that no matter how much time elapses the insect is never going to get to a point 2 metres from the start. It will get closer and closer but never quite reach that point. The distance of 200 centimetres is what is called a "sum to infinity".

If  $|r| < 1$ , as  $n$  gets larger and larger  $r^n$  approaches zero. So we get what is called the **sum to infinity** of the geometric series:

$$S_{\infty} = a + ar + ar^2 + \dots = \frac{a}{1 - r}.$$

**Example 8:** Psychologists have determined that each year you live seems shorter than the year before. They estimate that this year will only seem to be 99% as long as last year. If this is true, and you were to live forever, how long would this eternal life appear to be in terms of the apparent length of your first year of life?

**Solution:** Here we have a geometric series with  $a = 1$ ,  $r = 0.99$ .

$$S_{\infty} = \frac{1}{0.01} = 100.$$

**Example 9:** Under the assumptions in Example 9, assuming that you lived forever when would you appear to have lived for 50 years?

**Solution:** We want to find  $n$  so that  $S_n = 50$ .

$$\text{Hence } \frac{1 - 0.99^n}{.01} = 50.$$

$$\text{So } 1 - 0.99^n = 0.5.$$

$$\text{and hence } 0.99^n = 0.5.$$

By trial and error, and with the help of a calculator, the apparent midpoint of the life of someone who lives forever would appear to be at approximately age 69.



(Once we learn about logarithms we would be able to find  $n$  more easily, as  $n = \frac{\log 0.5}{\log 0.99}$   
 $= 68.967 \dots$  However the most appropriate answer would again be “approximately 69 years.)

**Example 10:** Assuming that you will live to be 100, when will your life appear to be half over?

**Solution:** The apparent length of your life, if you lived to be 100, would be 63.4 years, in terms of your first year of life. When you are 38 years old your apparent length of life would be 31.7 years so your life would appear to be half over!